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# The slow transverse motion of a flat plate through a non-diffusive stratified fluid

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We consider the two-dimensional flow produced by the slow horizontal motion of a vertical plate of height $2b$ through a vertically stratified ($\rho = \rho_0(1-\beta z)$) non-diffusive viscous fluid. Our results are valid when $U^2/\beta g b^2 \ll Ub/\nu \ll 1$, where $U$ is the speed of the plate and $\nu$ the kinematic viscosity of the fluid. Upstream of the body we find a blocking column of length $10^{-2}b^4/(U\nu/\beta g)$. This column is composed of cells of closed streamlines. The convergence of these cells near the tips of the plate leads to alternate jets. The plate itself is embedded in a vertical shear layer of thickness $(U\nu/\beta g)^{\frac{1}{3}}$ ($\ll b$). In the upstream portion of this layer the vertical velocities are of order $U$ and in the downstream portion of order $Ub/(U\nu/\beta g)^{\frac{1}{3}}$ ($\gg U$). The flow is uniform and undisturbed downstream of this layer.

## 1. Introduction

The flow produced by the motion of bodies through stratified fluids has excited some interest in recent years. The study of these flows has generally taken one of three theoretical approaches. The first approach considers the flow as inviscid and steady, takes into account non-linear effects, and is generally based on the use of Long's equation, for example, Long (1953). A nearly uniform upstream velocity profile is assumed and theoretical and experimental results are in good agreement. However, at very low internal Froude numbers one expects a blocking column upstream of the body, which, in the absence of viscosity, should extend to infinity; this conflicts with the assumption of a nearly uniform upstream velocity. The second approach considers the inviscid linearized initial-value problem, for example, Bretherton (1967). This enables us to study the formation of the blocking column. However, when viscosity is put into the problem it is assumed that the Prandtl number equals one, and while this gives a perfect analogy with a rotating problem it is not very appropriate for the stratified case, especially if the stratification is due to salinity variations. In this case, the Prandtl, or rather the Schmidt, number is of order 1000. The third approach considers the viscous low Reynolds number problem (Martin & Long 1968). This approach has considered the slow horizontal motion of horizontal plates and does not shed much light on the nature of blocking. This work uses the third approach but considers a vertical body and thus should reveal some aspects of the blocking column. The fluid is assumed non-diffusive, which seems reasonable for the high Schmidt number case.

## 2. Specification of the problem

We consider a flat plate of height $2b$ moving slowly to the left with speed $U$ through a linearly stratified non-diffusive fluid. The flow, as viewed from a Cartesian co-ordinate system attached to the plate, is steady. The plate is of

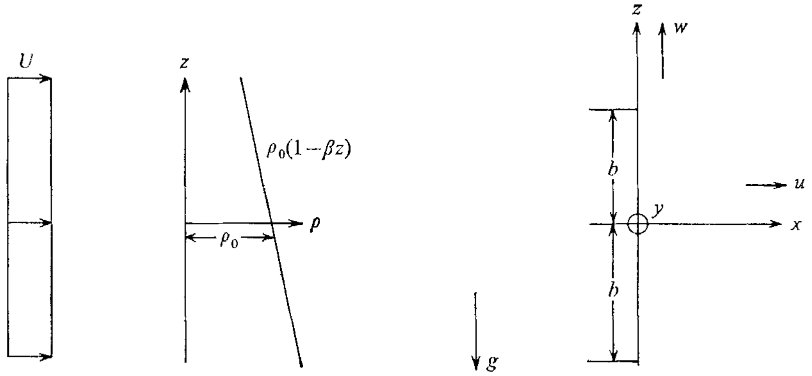


FIGURE 1. Geometry of the flow field.

infinite extent in the $y$ direction, see figure 1. The stratification is sufficiently small so that the Boussinesq approximation is valid, and $\nu$ is assumed constant. The governing equations are then

$$\rho_0 \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u, \tag{1a}$$

$$\rho_0 \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \rho g + \mu \nabla^2 w, \tag{1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1c}$$

$$D\rho/Dt = 0, \tag{1d}$$

where
$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{2}$$

We now introduce the stream function $\overline{\psi}$ by

$$u = -\partial\overline{\psi}/\partial z, \quad w = \partial\overline{\psi}/\partial x. \tag{3}$$

Integrating $(1d)$ leads to

$$\rho = \rho(\overline{\psi}). \tag{4a}$$

Differentiating $(1a)$ and $(1b)$ with respect to $z$ and $x$ respectively and subtracting the results yields

$$\rho_0 \frac{D}{Dt} \nabla^2 \overline{\psi} = -\frac{d\rho}{d\overline{\psi}} g \frac{\partial \overline{\psi}}{\partial x} + \mu \nabla^4 \overline{\psi}. \tag{4b}$$

The boundary conditions on (1) and (4) are as follows:

as $|z|$, $|x| \to \infty$, $\overline{\psi} \to -Uz$, $\rho \to \rho_0(1-\beta z)$; (5*a*)

and at $x = 0$, $|z| \leqslant 1$, $\partial\overline{\psi}/\partial z = \partial\overline{\psi}/\partial x = 0$. (5*b*)

We evaluate (4*a*) as $x \to -\infty$ and find that

$$\rho = \rho_0 + \rho_0 \frac{\beta}{U} \overline{\psi}. \tag{6}$$

We now non-dimensionalize as follows:

$$\left.\begin{aligned} (u', w') &\equiv (u, w)/U, \quad \overline{\psi}' \equiv \overline{\psi}/Ub, \\ (x', z') &\equiv (x, z)/b. \end{aligned}\right\} \tag{7}$$

Equation (4*b*) then becomes

$$\frac{U^2}{\beta g b^2} \frac{D'}{Dt'} \nabla'^2 \overline{\psi}' = -\frac{\partial \overline{\psi}'}{\partial x'} + \frac{U\nu}{\beta g b^3} \nabla'^4 \overline{\psi}'. \tag{8}$$

We now require that $U$ be sufficiently small so that

$$U^2/\beta g b^2 \ll U\nu/\beta g b^3. \tag{9}$$

This is exactly the same as requiring $Ub/\nu \ll 1$. We define a non-dimensionalized perturbation stream function, $\psi$, as follows:

$$\psi(x', z') \equiv \overline{\psi}' + z'. \tag{10}$$

Substituting this into (8) subject to condition (9) and dropping the primes on $(x', z')$, we obtain

$$\nabla^4 \psi - L^3 \, \partial\psi/\partial x = 0, \tag{11}$$

where $$L \equiv b/(U\nu/\beta g)^{\frac{1}{3}}.$$

The boundary conditions on $\psi$ are as follows:

as $|x|$, $|z| \to \infty$, $\psi \to 0$; (12*a*)

at $x = 0$, $|z| \leqslant 1$, $\psi = z$, $\partial\psi/\partial x = 0$. (12*b*)

We shall solve (11) subject to (12) in the following manner: Since $\psi(x, z)$ is an odd function of $z$, we introduce its Fourier sine transform. We then find two solutions for $\psi$, one valid for $x < 0$ and one for $x > 0$. We then assume $\psi(0, z)$ and $\partial\psi(0, z)/\partial x$ are unknown functions which are specified at $x = 0$, $|z| \geqslant 1$ in such a manner that the up- and downstream solutions for $\psi$ along with their first three partial derivatives with respect to $x$ are continuous for $x = 0$, $|z| \geqslant 1$. The governing equation itself indicates that all higher-order $x$ derivatives will then be continuous. This leads to a coupled set of linear integro-differential equations for the Hilbert transforms of $\psi(0, z)$, $\partial\psi(0, z)/\partial x$. We then limit our interest to cases where $L \gg 1$ and indicate an iterative procedure for the solution of this set. We can show that this set of equations only need be solved if we are interested in the flow near the tips of the plate. We exclude this region from consideration and compute the flow everywhere else.

## 3. Solution of the problem

The following functions and relationships will be used in the solution of this problem:

$$\bar{F}(K,x) \equiv (2/\pi)^{\frac{1}{2}} \int_0^\infty \psi(x,z) \sin Kz \, dK, \tag{13a}$$

$$\psi(x,z) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \bar{F}(K,x) \sin Kz \, dK, \tag{13b}$$

$$\bar{F}(K) \equiv \bar{F}(K,0), \tag{13c}$$

$$\bar{G}(K) \equiv [\partial \bar{F}(K,x)/\partial x]|_{x=0} \tag{13d}$$

$$\bar{G}(K) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \frac{\partial \psi}{\partial x}(0,z) \sin Kz \, dK, \tag{13e}$$

$$F(z) \equiv (2/\pi)^{\frac{1}{2}} \int_0^\infty \bar{F}(K) \cos Kz \, dK, \tag{13f}$$

$$G(z) \equiv (2/\pi)^{\frac{1}{2}} \int_0^\infty \bar{G}(K) \cos Kz \, dK, \tag{13g}$$

$$F(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\psi(0,z')}{z'-z} dz', \tag{13h}$$

$$\psi(0,z) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F(z')}{z'-z} dz', \tag{13i}$$

$$G(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\partial \psi(0,z')/\partial x}{z'-z} dz', \tag{13j}$$

$$\frac{\partial \psi}{\partial x}(0,z) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{G(z')}{z'-z} dz'. \tag{13k}$$

The integrals in (13 *h*–*k*) are principal values. $F(z)$ and $G(z)$ are the Hilbert transforms of $\psi(0,z)$ and $\partial\psi(0,z)/\partial x$ respectively. Titchmarsh (1937, p. 119) shows that (13 *h*–*k*) follow from (13 *f*–*g*).

Substituting (13 *b*) into (11) leads to the following solution for $\bar{F}(K,x)$:

for $x > 0$
$$\bar{F}(K,x) = A_1(K) e^{\alpha_1(K)x} + A_2(K) e^{\alpha_2(K)x}; \tag{14a}$$

and for $x < 0$
$$\bar{F}(K,x) = A_3(K) e^{\alpha_3(K)x} + A_4(K) e^{\alpha_4(K)x}. \tag{14b}$$

The $\alpha(K)$'s are the roots of the polynomial

$$\alpha^4 - 2K^2\alpha^2 - L^3\alpha + K^4 = 0. \tag{15}$$

$\alpha_1$ and $\alpha_2$ are complex conjugates for all real $K$ and have negative real parts which become more negative as $K$ increases. $\alpha_4$ and $\alpha_3$ are real for all real $K$ and increase monotonically with $K$. For a more complete discussion of these roots, see Janowitz (1968). As $K/L \to 0$ we can show

$$\alpha_1 \approx e^{\frac{2}{3}\pi i} L, \quad \alpha_2 \approx e^{-\frac{2}{3}\pi i} L, \quad \alpha_3 \approx K^4/L^3, \quad \alpha_4 \approx L. \tag{16}$$

For the up- and downstream solutions for $\psi(0,z)$ and $\partial\psi(0,z)/\partial x$ to be equal at $x = 0$, we require:

$$A_{1,3}(K) = \frac{\alpha_{2,4}\bar{F}(K)-\bar{G}(K)}{\alpha_{2,4}-\alpha_{1,3}}, \tag{17a}$$

$$A_{2,4}(K) = \frac{\bar{G}(K)-\alpha_{1,3}\bar{F}(K)}{\alpha_{2,4}-\alpha_{1,3}}. \tag{17b}$$

For the up- and downstream solutions for $\partial^2\psi/\partial x^2$ and $\partial^3\psi/\partial x^3$ to be equal for $x = 0$, $|z| \geqslant 1$, we obtain after much algebraic manipulation:

$$\text{for } |z| \geqslant 1 \quad 4\frac{\partial^2 G(z)}{\partial z^2} - \tfrac{1}{2}L^3 F(z) = (2/\pi)^{\frac{1}{2}} \int_0^\infty (I_1\bar{G}(K)+I_2\bar{F}(K))\cos Kz\, dK, \tag{18a}$$

$$4\frac{\partial^4 F(z)}{\partial z^4} - \tfrac{1}{2}L^3 G(z) = (2/\pi)^{\frac{1}{2}} \int_0^\infty (I_2\bar{G}(K)+I_3\bar{F}(K))\cos Kz\, dK, \tag{18b}$$

where

$$I_1(K) = 2[\theta(K)]^{\frac{1}{2}} K - 4K^2, \tag{18c}$$

$$I_2(K) = \frac{L^3 K}{[\theta(K)]^{\frac{1}{2}}} - \tfrac{1}{2}L^3, \tag{18d}$$

$$I_3(K) = 4K^4 - K[\theta(K)]^{\frac{1}{2}}(\theta(K)-2K^2), \tag{18e}$$

$$\theta(K) = \frac{4K^2}{3} + 2^{-\frac{1}{3}}\left[L^6 + \frac{2^7}{27}K^6 + L^3\left(L^6 + \frac{2^8K^6}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + 2^{-\frac{1}{3}}\left[L^6 + \frac{2^7}{27}K^6 - L^3\left(L^6 + \frac{2^8K^6}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}. \tag{18f}$$

For $x = 0$, $|z| \leqslant 1$ we must satisfy (5*b*), i.e.

$$\psi(0,z) = z, \quad \partial\psi(0,z)/\partial x = 0.$$

Substituting these conditions into (13*i*, *k*) yields for $|z| \leqslant 1$

$$z = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F(z')}{z'-z}\,dz', \tag{19a}$$

$$0 = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{G(z')}{z'-z}\,dz'. \tag{19b}$$

We now solve for $F(z)$ and $G(z)$ for $|z| \leqslant 1$ as functions of $F(z)$ and $G(z)$ for $|z| \geqslant 1$. In terms of these functions, we obtain, after some algebraic manipulations (see Tricomi 1957, pp. 173–185) for $|z| \leqslant 1$,

$$F(z) = \frac{\bar{C}-z^2}{(1-z^2)^{\frac{1}{2}}} - \frac{1}{\pi}\int_1^\infty \frac{2t(t^2-1)^{\frac{1}{2}}F(t)}{(1-z^2)^{\frac{1}{2}}(t^2-z^2)}\,dt, \tag{20a}$$

$$G(z) = \frac{\bar{D}}{(1-z^2)^{\frac{1}{2}}} - \frac{1}{\pi}\int_1^\infty \frac{2t(t^2-1)^{\frac{1}{2}}G(t)}{(1-z^2)^{\frac{1}{2}}(t^2-z^2)}\,dt, \tag{20b}$$

where $\bar{C}$ and $\bar{D}$ are, thus far, arbitrary constants. We now substitute (20) into (13*i*, *k*); we obtain for $|z| \leqslant 1$, as required,

$$\psi(0,z) = z, \quad \partial\psi(0,z)/\partial x = 0;$$

and for $|z| \geqslant 1$

$$\psi(0,z) = z-(z^2-1)^{\frac{1}{2}}+\frac{1-\bar{C}}{(z^2-1)^{\frac{1}{2}}}+\frac{1}{\pi}\frac{1}{(z^2-1)^{\frac{1}{2}}}\int_1^\infty \frac{2t(t^2-1)^{\frac{1}{2}}}{t^2-z^2}F(t)\,dt, \tag{21a}$$

$$\frac{\partial\psi}{\partial x}(0,z) = \frac{1-\bar{D}}{(z^2-1)^{\frac{1}{2}}}+\frac{1}{\pi(z^2-1)^{\frac{1}{2}}}\int_1^\infty \frac{2t(t^2-1)^{\frac{1}{2}}}{t^2-z^2}G(t)\,dt. \tag{21b}$$

Let us briefly summarize what remains to be accomplished. If we obtain $F(z)$ and $G(z)$ for $|z| \geqslant 1$ we can obtain $\psi(0,z)$, $\partial\psi(0,z)/\partial x$ from (21). From (13$a, e$) we obtain $\bar{F}(K)$, $\bar{G}(K)$. Thence, from (17) and (14), we obtain $\bar{F}(K,x)$, and from (13$b$) $\psi(x,z)$. We specify certain conditions on $\psi(0,z)$ and $\partial\psi(0,z)/\partial x$ as $z$ approaches 1·0 from above ($1^+$). We first require that $\psi(0,1^+) = 1$ and $\partial\psi(0,1^+)/\partial x = 0$. If we integrate the integrals in (21) by parts (letting

$$u = F(t), \quad dv = dt\,2t(t^2-1)^{\frac{1}{2}}/(t^2-z^2), \quad \text{etc.})$$

and require that
$$1-\bar{C} = \frac{2}{\pi}\int_1^\infty (t^2-1)^{\frac{1}{2}}\frac{dF}{dt}(t)\,dt, \tag{22a}$$

and
$$1-\bar{D} = \frac{2}{\pi}\int_1^\infty (t^2-1)^{\frac{1}{2}}\frac{dG}{dt}(t)\,dt, \tag{22b}$$

we can show that these conditions are satisfied. Differentiating the result for $\psi(0,z)$ and $\partial\psi(0,z)/\partial x$ with respect to $z$ leads to

$$\frac{\partial\psi}{\partial z}(0,z) = 1-\frac{z}{(z^2-1)^{\frac{1}{2}}}+\frac{2z}{\pi(z^2-1)^{\frac{1}{2}}}\int_1^\infty \frac{(t^2-1)^{\frac{1}{2}}}{t^2-z^2}\frac{dF}{dt}dt \tag{23a}$$

and
$$\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial x}(0,z)\right) = \frac{2z}{\pi(z^2-1)^{\frac{1}{2}}}\int_1^\infty \frac{(t^2-1)^{\frac{1}{2}}}{t^2-z^2}\frac{dG}{dt}dt. \tag{23b}$$

We now require that $\partial\psi/\partial z|_{z=1^+} = 1$ and $\partial(\partial\psi/\partial x)/\partial z|_{z=1^+}$ remain finite.

If we now integrate the integrals in (23) by parts and require that

$$\int_1^\infty \frac{d}{dt}\left(\frac{1}{2t}\frac{dF}{dt}\right)(t^2-1)^{\frac{1}{2}}\,dt = -\tfrac{1}{4}\pi \tag{24a}$$

and
$$\int_1^\infty \frac{d}{dt}\left(\frac{1}{2t}\frac{dG}{dt}\right)(t^2-1)^{\frac{1}{2}}\,dt = 0, \tag{24b}$$

we can show that these conditions are satisfied. Thus, as $z \to 1^+$, $\bar{\psi}' \to 0$, $u' \to 0$, $w' \to 0$ and $\partial w'/\partial z$ remains finite. As our last condition, we require that $\partial u'(0,z)/\partial z|_{z=1^+}$ remain finite. Proceeding as before, we find

$$\int_1^\infty \frac{d}{dt}\left(\frac{1}{2t}\frac{d}{dt}\left(\frac{1}{2t}\frac{dF}{dt}\right)\right)(t^2-1)^{\frac{1}{2}}\,dt = 0. \tag{24c}$$

We also require that $F(z)$ and $G(z)$ remain finite for large $z$. Equation (24) then determines the solutions of (18) uniquely.

## 4. Solution for large $L$

We now restrict ourselves to the case where $L \gg 1$. This means that the vertical dimension of the plate is much larger than $(U\nu/\beta g)^{\frac{1}{3}}$, which we shall see is the thickness of the shear layers occurring in the fluid. We have already restricted ourselves to the case where $Ub/\nu \ll 1$. Now requiring $L \gg 1$ implies

$$U^2/\beta g b^2 \ll Ub/\nu \ll 1.$$

We can solve (18) subject to (24) using (21) by the following iterative procedure. First neglect the right-hand side of (18$a, b$) and solve the resulting equations subject to (24). This yields first approximations to $G(z)$ and $F(z)$, say $F_0(z)$, $G_0(z)$, and we find

$$F_0(z) = O(1/L^{\frac{1}{2}}), \quad G_0(z) = O(L^{\frac{1}{2}}),$$

both functions vanishing for $L\,|\,|z|-1| \gtrsim 10$. With these functions we compute $\psi_0(0,z)$, $\partial\psi_0(0,z)/\partial x$, $\bar{F}_0(K)$, and $\bar{G}_0(K)$ from (21) and (13$a, e$). We can now compute the right-hand side of (18$a, b$) with $\bar{F}_0(K)$, $\bar{G}_0(K)$ and then solve (18$a, b$) subject to (24) with this right-hand side. We now find $F_1(z) = O(1/L^{\frac{1}{2}})$ and $G_1(z) = O(L^{\frac{1}{2}})$ within the layer $L\,|\,|z|-1| = O(1)$ and for $|z|-1 = O(1)$, $F_1(z) = O(1/L)$, $G_1(z) = O(1)$. We can recompute the right-hand side of (18$a, b$) and go through the iteration again and find that the $F(z)$ and $G(z)$ retain the same orders of $L$ (to lowest order) as before, only the numerical values changing. We will show that the numerical values of $F(z)$ and $G(z)$ determine $\psi(x,z)$ only in the immediate vicinity of the tips of the plate, a region we now exclude from consideration. We now summarize.

For $L\,|\,|z|-1| = O(1)$,

$$\left.\begin{aligned} F(z) = O(1/L^{\frac{1}{2}}), \quad G(z) = O(L^{\frac{1}{2}}),\\ \bar{\psi}'(0,z) = -z+\psi(0,z) = O(1/L^{\frac{1}{2}}),\\ \partial\bar{\psi}'(0,z)/\partial x = O(L^{\frac{1}{2}}), \quad u' = -\partial\bar{\psi}'/\partial z = O(L^{\frac{1}{2}}).\end{aligned}\right\} \tag{25a}$$

For $|\,|z|-1| = O(1)$,

$$\left.\begin{aligned} F(z) = O(1/L), \quad G(z) = O(1),\\ \psi(0,z) = z-(z^2-1)^{\frac{1}{2}}+O(1/L),\\ \partial\psi(0,z)/\partial x = O(1),\\ u' = -\partial\bar{\psi}'/\partial z = (z/(z^2-1)^{\frac{1}{2}})+O(1/L).\end{aligned}\right\} \tag{25b}$$

We now show that this is sufficient information to determine $\psi(x,z)$ away from the tips of the plate.

## 5. The downstream solution

Using (13$b$), (14$a$) and (17) we may write for $x \geqslant 0$,

$$\psi(x,z) = (2/\pi)^{\frac{1}{2}}\int_0^\infty \frac{\alpha_2\bar{F}\,e^{\alpha_1 x}-\alpha_1\bar{F}\,e^{\alpha_2 x}}{\alpha_2-\alpha_1}\sin Kz\,dK$$

$$+(2/\pi)^{\frac{1}{2}}\int_0^\infty \frac{\bar{G}\,e^{\alpha_2 x}-\bar{G}\,e^{\alpha_1 x}}{\alpha_2-\alpha_1}\sin Kz\,dK, \tag{26a}$$

where
$$\left.\begin{aligned}\alpha_1(K) &= e^{\frac{2}{3}\pi i}L(1+O((K/L)^2)),\\ \alpha_2(K) &= e^{-\frac{2}{3}\pi i}L(1+O((K/L)^2)),\end{aligned}\right\} \qquad (26b)$$

and
$$\bar{F}(K) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\left[\frac{J_1(K)}{K}+O\left(\frac{1}{L}\right)\right],$$
$$\bar{G}(K) = O(1). \qquad (26c)$$

$(\frac{1}{2}\pi)^{\frac{1}{2}}J_1(K)/K$ is the Fourier sine transform of
$$\begin{cases} z-(z^2-1)^{\frac{1}{2}} & (|z| \geqslant 1),\\ z & (|z| \leqslant 1).\end{cases}$$

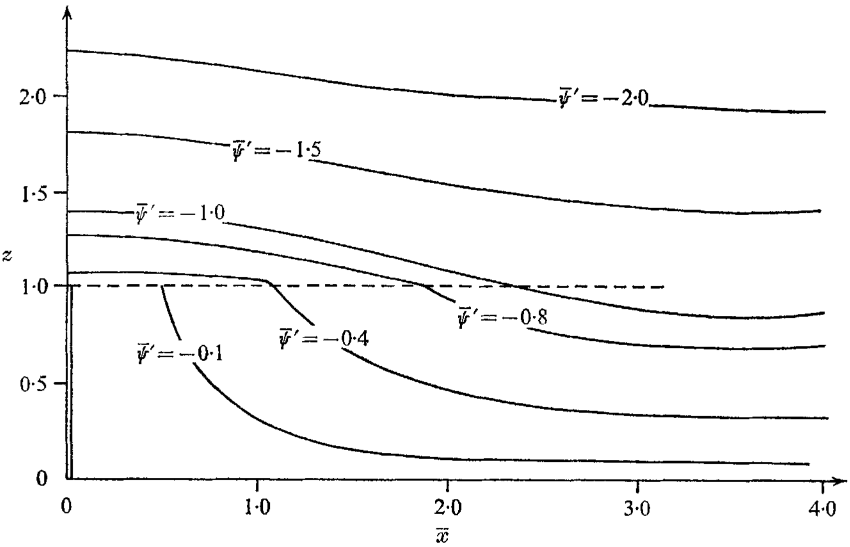


FIGURE 2. The streamline pattern downstream of the plate. Dashed line indicates position of horizontal shear layer.

The contributions to $\psi(x,z)$ outside the shear layer $||z|-1| = O(1/L)$ come from the Fourier transforms in the region $K = O(1)$. In this region we may take
$$\alpha_1 = e^{\frac{2}{3}\pi i}L, \quad \alpha_2 = e^{-\frac{2}{3}\pi i}L.$$
The flow in the shear layer is determined by behaviour of the Fourier transform for $K = O(L)$. To determine this behaviour would require an explicit solution for $F(z)$, $G(z)$. Since we can understand the features of the flow without these details, we do not solve for them. Hence, from (26*a*), for $x \geqslant 0$ and $L\,||z|-1| \gg 1$,
$$\bar{\psi}'(\bar{x},z) = -z+(2/3^{\frac{1}{2}})e^{-\frac{1}{2}\bar{x}}\sin(\tfrac{1}{2}\,3^{\frac{1}{2}}\bar{x}+\tfrac{1}{3}\pi)\begin{cases} z-(z^2-1)^{\frac{1}{2}} & (|z| \geqslant 1)\\ z & (|z| \leqslant 1)\end{cases}+O(1/L), \quad (27)$$
where $\bar{x} \equiv Lx$. The solution reveals a vertical shear layer of thickness $(U\nu/\beta g)^{\frac{1}{3}}$ on the downstream side of the plate. The vertical velocities in this layer $(\partial\bar{\psi}/\partial x)$ are order $L$ and downstream of this layer $\bar{\psi}' = -z$, or uniform flow exists. Some streamlines are plotted in figure 2. The horizontal dashed line denotes the shear layer. We note that the streamlines oscillate slightly about their equilibrium heights. We also note for $|z| \geqslant 3$ the streamlines are disturbed by less than 5% of the equilibrium heights for all $\bar{x}$.

## 6. The upstream solution

Using (13*b*), (14*b*) and (17) we can write for $x \leqslant 0$

$$\psi(x,z) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \frac{\alpha_4 e^{\alpha_3 x} - \alpha_3 e^{\alpha_4 x}}{\alpha_4 - \alpha_3} \bar{F}(K) \sin Kz \, dK$$

$$+ (2/\pi)^{\frac{1}{2}} \int_0^\infty \frac{e^{\alpha_4 x} - e^{\alpha_3 x}}{\alpha_4 - \alpha_3} \bar{G}(K) \sin Kz \, dK, \tag{28a}$$

where

$$\left.\begin{aligned} \alpha_3 &= (K^4/L^3)(1 + O((K/L)^4)), \\ \alpha_4 &= L(1 + O((K/L)^2)). \end{aligned}\right\} \tag{28b}$$

and

$$\left.\begin{aligned} \bar{F}(K) &= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \left[\frac{J_1(K)}{K} + O\left(\frac{1}{L}\right)\right], \\ \bar{G}(K) &= O(1). \end{aligned}\right\} \tag{28c}$$

For $||z|-1| = O(1)$ and $|Lx| = O(1)$, the thin vertical shear layer upstream and adjacent to the plate, the contributions to $\psi(x,z)$ come from the Fourier transforms in the range $K = O(1)$. Here we may take $\alpha_3 = 0$, $\alpha_4 = L$ and we obtain

$$\begin{aligned} \bar{\psi}'(x,z) &= -z + \psi(x,z) \\ &= -(z^2-1)^{\frac{1}{2}} + O(1/L) \quad (|z| \geqslant 1), \\ &= O(1/L) \quad\quad (|z| \leqslant 1). \end{aligned} \tag{29}$$

For $L\,||z|-1| = O(1)$, explicit solutions for $F(z)$ and $G(z)$ are again required and we again exclude this area from attention. Now consider the region upstream of this thin vertical shear layer, i.e. $|Lx| \gg 1$.

Then $\alpha_4|x| = O(Lx) \gg 1$ and we can neglect the contributions to $\psi(x,z)$ from the coefficients of $e^{\alpha_4 x}$. Now $\alpha_3 x \cong (K^4/L^3)x = (K/L)^4(Lx)$. Hence, contributions to $\psi(x,z)$ when $K = O(L)$ are multiplied by $e^{\alpha_3 x} \cong e^{-|Lx|}$ and are exponentially small. So we may therefore take

$$\psi(x,z) = \int_0^\infty \frac{J_1(K)}{K} e^{K^4 x/L^3} \sin Kz \, dK \tag{30}$$

for all $z$ for $|Lx| \gg 1$. The streamline pattern computed from (10) and (30) is shown in figure 3 with the horizontal axis taken as $|x/L^3|^{\frac{1}{4}}$. We do not compute $\bar{\psi}'(x,z)$ for $|z| < 1$, $|x/L^3|^{\frac{1}{4}} < 0{\cdot}03$ since the computed values of $\bar{\psi}' \approx 10^{-4}$ are on the order of the error in Simpson's rule evaluation of (30).

The curves for $\bar{\psi}' = 0$ and $\bar{\psi}' = -1$ are shown together with some other information.

First we see a blocking column in front of the plate extending to a distance of $|x/L^3|^{\frac{1}{4}} = 0{\cdot}360$. Outside this region $u' > 0$ and the fluid particles all move in downstream direction. The blocking column itself is composed of cells of closed streamlines. The boundaries of these cells, i.e. the curves $\bar{\psi}' = 0$, and the curve $\bar{\psi}' = +0{\cdot}030$ were computed from (30). The direction of the motion in the other two cells is indicated. These cells converge towards the tip of the plate. Near the tip of the plate the motion in these cells would show up as horizontal alternating jets. The maximum calculated value of $\bar{\psi}'$ in the blocking column is approximately $+0{\cdot}05$, indicating that the density variation in the column is only $1/20$ of that occurring over the same range of $z$, upstream of the column. The $\bar{\psi}' = -1$ streamline is also plotted. The dashed line in this figure is the locus of maximum

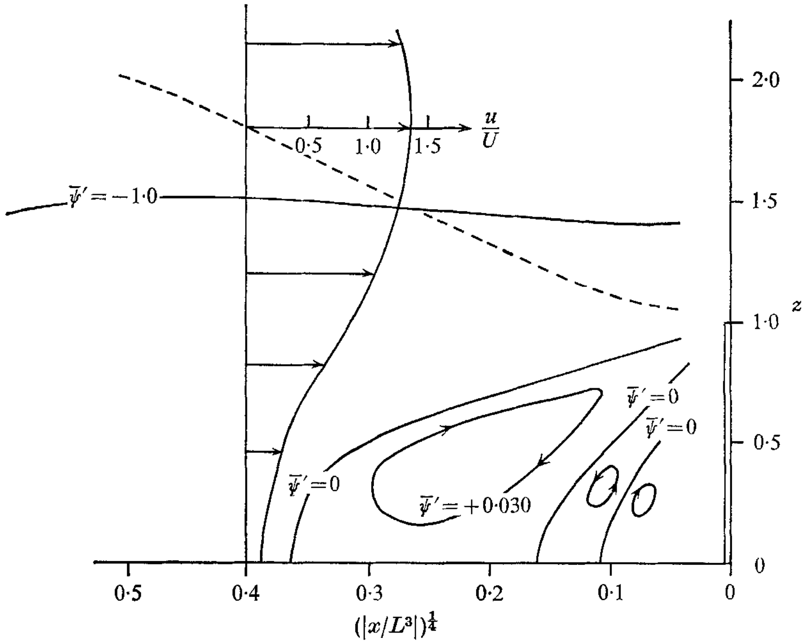


FIGURE 3. The streamline pattern upstream of the plate. Dashed line is locus of maximum horizontal velocity. Horizontal velocity at $|x/L^3|^{\frac{1}{4}} = 0{\cdot}40$ is also shown.

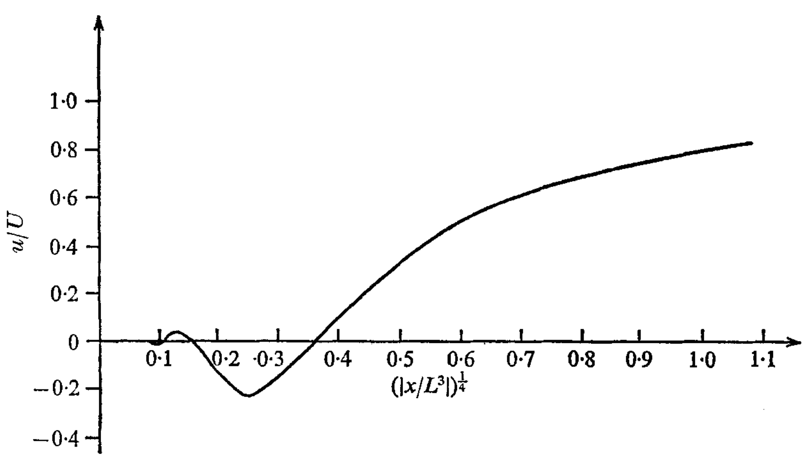


FIGURE 4. The upstream horizontal velocity at $z = 0$.

horizontal velocity outside the blocking column. We also have plotted the horizontal velocity profile at $|x/L^3|^{\frac{1}{4}} = 0{\cdot}40$. The maximum velocity at this station is $1{\cdot}3U$. In figure 4 the horizontal velocity at $z = 0$ is plotted versus $|x/L^3|^{\frac{1}{4}}$. Upstream of $|x/L^3|^{\frac{1}{4}} = 0{\cdot}360$ the velocity increases monotonically towards $1{\cdot}0$.

## 7. Conclusion

Thus, the non-diffusive solution paints an interesting picture of the flow including horizontal and vertical shear layers of thickness $(U\nu/\beta g)^{\frac{1}{3}}$ as well) as a blocking column composed of cells of closed streamlines and uniform flow slightly downstream of the body.

An important question concerning the nature of the blocking column remains. which cannot be answered within the context of the non-diffusive solution. We evaluated $\rho(\overline{\psi}')$ far upstream of the body, the only place we could do this. Our solution then shows a blocking column composed of cells of streamlines which do not originate at infinity upstream. Thus, we cannot show that (6) describes the relationship between $\rho$ and $\overline{\psi}'$ in the column. What is worse, closed streamlines imply overturning which may lead to a breakdown to turbulence.

This impasse is similar to the situation which arises when Long's (1953) equation is used to study the stratified flow over obstacles in a channel at low internal Froude numbers (based on the channel height). There, as here, the density is related to the stream function at infinity upstream and (unstable) regions of closed streamlines are predicted. Experimentally, we observe a breakdown to turbulence in the unstable regions and the flow as predicted outside of these regions. This may occur in our case.

More probably, due to the small velocities, the small density fluctuations, and to the presence of stagnation points predicted in the blocking column, diffusive effects are important there. The flow field within the column should change, possibly eliminating the unstable density configuration, and producing a more uniform density distribution. Outside of the column the effects of diffusion should remain small and the flow field should agree with that predicted here as long as $U^2/\beta g b^2 \ll Ub/\nu \ll 1$. Of course, we must await the (difficult) diffusive solution to substantiate this conclusion.

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